Global Bifurcation In Four-Component Bose-Einstein Condensates In Space

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Energy Function

The BEC comprises 4 gas components, each described by its own wave function $u_j : \Omega \to \mathbb{R}$ ($j = 1, 2, 3, 4$), where Ω is unit ball in $\mathbb{R}^3$. The energy is given by:

$$E_\mu(u) = \frac{1}{2} \int_\Omega \sum_{j=1}^{4} \left(|\nabla u_j|^2 - \mu u_j^2 + \frac{g}{2} u_j^4 \right) + \sum_{i,j=1 (i \neq j)}^{4} \frac{\tilde{g}}{2} u_i^2 u_j^2 \, dx. \tag{1}$$

where

- $u = (u_1, \dots, u_4) : \Omega \to \mathbb{R}^4$ represents the vector of components.
- The constant μ correspond to the chemical potentials.
- The coupling constant $\tilde{g}$ describe the interaction between the i-th and j-th components.
- g is the energy coefficient of j-th component itself.

Gross-Pitaevskii Equation

The solutions of BECs are the solution to

$$\nabla_u E_\mu(u) = 0, \text{ with Neumann boundary conditions.} \tag{2}$$

It can be explicitly written as (the Gross-Pitaevskii equation):

$$-\Delta u_j - \mu u_j + g u_j^3 + \sum_{i=1 (i \neq j)}^{4} \tilde{g} u_i^2 u_j = 0, \quad \frac{\partial u_j}{\partial x}\bigg|_{|x|=1} = 0, \; j = 1, 2, 3, 4. \tag{3}$$

System (3) has a branch of trivial solution given by

$$u_\mu = (c_\mu, c_\mu, c_\mu, c_\mu), \quad c_\mu = \Big(\mu/(g + 3\tilde{g})\Big)^{1/2}, \tag{4}$$

The purpose of this work is to explore existence of bifurcation of non-trivial solutions arising from u_μ, using bifurcation parameter $\mu \in \mathbb{R}$.

Functional Space Reformulation

We define the following notations in the context of our bifurcation problem.

- The space where the solutions to the problem (2) live is given by

$$\mathcal{X} := \left\{ u \in H^2(\Omega; \mathbb{R}^4) : \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} = 0 \right\} \tag{5}$$

- The energy functional $E_\mu : \mathcal{X} \to \mathbb{R}$ is given by formula (1).
- The unbounded operator $\mathcal{L} := -\Delta + I : \mathcal{X} \to L^2(\Omega, \mathbb{R}^4)$ is a closed self-adjoint Fredholm operator of index zero, and $\mathcal{L}^{-1} : L^2(\Omega, \mathbb{R}^4) \to \mathcal{X}$ is continuous.
- Operator $K_\mu : \mathcal{X} \to \mathcal{X} \subset L^2(\Omega, \mathbb{R}^4)$, where

$$K_\mu(u) = (K_1(\mu, u), ..., K_4(\mu, u)),$$

$$K_j(\mu, u) = -(\mu + 1)u_j + gu_j^3 + \tilde{g} \sum_{i=1 (i \neq j)}^{4} u_i^2 u_j.$$

  K_μ is well-defined and continuous.

Then the gradient $\nabla_u E_\mu : \mathcal{X} \to \mathcal{X}$ given by

$$\nabla_u E_\mu(u) = u + \mathcal{L}^{-1} K_\mu(u) \tag{gE}$$

is completely continuous field.

Group

Consider the action of the group

$$G = O(3) \times S_4$$

on the space $\mathscr{X}$ given by the formula

$$\begin{cases} \gamma u(x) = u(\gamma^{-1}x), & \gamma \in O(3), \\ \sigma u(x) = (u_{\sigma^{-1}(1)}, \dots, u_{\sigma^{-1}(4)}), & \sigma \in S_4. \end{cases}$$

where S_4 is the group of permutations of $\{1, 2, 3, 4\}$

Obviously $\mathscr{X}$ is a Hilbert G-representation with respect to the above action.

Remark: Since the energy $E_\mu(u)$ is G-invariant, its gradient $\nabla_u E_\mu(u)$ is G-equivariant completely continuous field, and consequently, the gradient G-equivariant degree theory can be applied to the bifurcation problem (2).

Method of Equivariant Degree Theory

- Set of trivial solutions by $\mathcal{T}$, i.e.
$$\mathcal{T} = \{(u_\mu, \mu); \mu \in \mathbb{R}\} \subset \mathscr{X} \times \mathbb{R}.$$
- Λ the critical set of μ such that the Hessian $\nabla^2 E_\mu(u_\mu)$ is not an isomorphism.
- Assume $\mu_0 \in \Lambda$ and $\mu_- < \mu_0 < \mu_+$ such that $[\mu_-, \mu_+] \cap \Lambda = \{\mu_0\}$.
- Let $\mathcal{U} \subset \mathscr{X}$ be sufficiently small neighbourhood of u_μ, then G-equivariant gradient degree $\nabla_G\text{-deg}(\nabla_u E_{\mu_\pm}(u_{\mu_\pm}), \mathcal{U})$ is well-defined.
- Define bifurcation index (the equivariant topological invariant) by the formula
$$\omega_G(\mu_0) := \nabla_G\text{-deg}\,(\nabla_u E_{\mu_-}(u_{\mu_-}), \mathcal{U}) - \nabla_G\text{-deg}\,(\nabla_u E_{\mu_+}(u_{\mu_+}), \mathcal{U}). \tag{6}$$

Theorem

If $\omega_G(\mu_0) \neq 0 \in U(G)$, then a global bifurcation from (u_{μ_0}, μ_0) occurs. Moreover, for every non-zero coefficient m_j in
$$\omega_G(\mu_0) = m_1(H_1) + m_2(H_2) + \dots m_r(H_r),$$
there exists a global family of non-trivial solutions with symmetries at least H_j. If (H_j) is a maximal orbit type then this family has exact symmetries (H_j).

Laplace Operator

Let $-\Delta : X \to L^2(\Omega;\mathbb{R})$ with Neumann boundary conditions, where $X = \left\{ u \in H^2(\Omega;\mathbb{R}) : \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \right\}$.

- The spectrum of the Laplace operator is given by

$$\boxed{\sigma(-\Delta) = \left\{ s_{km}^2 : k, m \in \mathbb{N} \right\}}, \tag{7}$$

  - * s_{km}^2 is the m-th positive zero of the function:

$$\psi_k(\lambda) := \left.\frac{d}{dr}\left(r^{-\frac{1}{2}} J_{k+\frac{1}{2}}(\lambda r)\right)\right|_{r=1} \tag{8}$$

  - * $J_{k+\frac{1}{2}}$ is the $(k+\frac{1}{2})$-th Bessel function of the first kind.

- The eigenspace associated to each s_{km}^2 is given by

$$\boxed{\mathcal{E}_{km} = \left\langle r^{-\frac{1}{2}} J_{k+\frac{1}{2}}(s_{km} r) T_k^n(\theta,\varphi) : 0 \le n \le k \right\rangle}, \tag{9}$$

  - * $T_k^n(\theta,\varphi)$ are called *surface harmonics* of degree k.
  - * Eigenspace $\mathcal{E}_{00} = \langle 1 \rangle$, corresponding to eigenvalue $S_{00} = 0$.

- The $O(3)$-irreducible representation $\mathcal{V}_k$, $k = 0, 1, \ldots$ is of dimension $2k+1$, and space $\mathcal{E}_{km}$ is $O(3)$-equivalent to $\mathcal{V}_k$.

Spectrum of Hessian

- The Hessian $\nabla^2 E_\mu(u_\mu) : \mathcal{X} \to \mathcal{X}$ is the linear operator

$$\nabla^2 E_\mu(u_\mu) = L^{-1}\left(-\Delta I + 2c_\mu^2 M\right), \tag{10}$$

where

$$M = \begin{pmatrix} g & \tilde{g} & \tilde{g} & \tilde{g} \\ \tilde{g} & g & \tilde{g} & \tilde{g} \\ \tilde{g} & \tilde{g} & g & \tilde{g} \\ \tilde{g} & \tilde{g} & \tilde{g} & g \end{pmatrix},$$

- The eigenvalues of matrix M are given by $g - \tilde{g}$ with multiplicity 3 and $g + 3\tilde{g}$ with multiplicity 1.
- The spectrum of $\nabla^2 E_\mu(u_\mu)$ consists of

$$\sigma(\nabla^2 E_\mu(u_\mu)) = \left\{ \begin{array}{ll} \frac{s_{km}^2 + 2c_\mu^2(g-\tilde{g})}{s_{km}^2+1} \text{ mult } 3(2k+1) \\ \frac{s_{km}^2 + 2c_\mu^2(g+3\tilde{g})}{s_{km}^2+1} \text{ mult } (2k+1) \end{array} : k, m \in \mathbb{N} \right\}. \tag{11}$$

- Under the immisible assumption $g \leq \tilde{g}$, the critical value is given by

$$\boxed{\mu_{km} := \frac{g+3\tilde{g}}{2(\tilde{g}-g)} s_{km}^2, \quad (k,m) \in \mathbb{N}^2.} \tag{12}$$

Isotypic Decomposition

Notice that:

- For space $\mathcal{X}$ in equation (5), we have

$$\mathcal{X} = \overline{\bigoplus_{k=0}^{\infty} (\mathcal{E}_{km})^4}, \tag{13}$$

  where $\mathcal{E}_{km}$ is the eigenspace represented in equation (9).

- $(\mathcal{E}_{km})^4 = \mathcal{E}_{km} \otimes \mathbb{R}^4$.
- $\mathbb{R}^4$ decompose in the S_4 irreducible representation $\mathcal{W}_0$ and $\mathcal{W}_4$, i.e. $\mathbb{R}^4 = \mathcal{W}_0 \oplus \mathcal{W}_4$.

The action of S_4 has the following characters for irreducible representations

Rep.	Character	(1)	(1,2)	(1,2)(3,4)	(1,2,3)	(1,2,3,4)
$\mathcal{W}_0$	χ_0	1	1	1	1	1
$\mathcal{W}_1$	χ_1	1	-1	1	1	-1
$\mathcal{W}_2$	χ_2	2	0	2	-1	0
$\mathcal{W}_3$	χ_3	3	-1	-1	0	1
$\mathcal{W}_4$	χ_4	3	1	-1	0	-1
$\mathbb{R}^4$	$\chi_{\mathbb{R}^4}$	4	2	0	1	0

Isotypic Decomposition

We conclude that:

- $(\mathcal{E}_{km})^4$ has the following decomposition

$$(\mathcal{E}_{km})^4 = (\mathcal{E}_{km} \otimes \mathcal{W}_0) \oplus (\mathcal{E}_{km} \otimes \mathcal{W}_4). \tag{14}$$

- Therefore, space $\mathcal{X}$ in equation (5) has the isotypic decomposition given by

$$\mathcal{X} = \overline{\bigoplus_{k=0}^{\infty} \left(\mathcal{X}_k^0 \oplus \mathcal{X}_k^4\right)}, \tag{15}$$

  where

$$\mathcal{X}_k^0 = \overline{\bigoplus_{m=1}^{\infty} \mathcal{E}_{km} \otimes \mathcal{W}_0}\ , \quad \mathcal{X}_k^4 = \overline{\bigoplus_{m=1}^{\infty} \mathcal{E}_{km} \otimes \mathcal{W}_4}\ . \tag{16}$$

- The isotypic component $\mathcal{X}_k^4$ is modeled on the $O(3) \times S_4$-irreducible representation in $\mathcal{V}_k \otimes \mathcal{W}_4$, which is of dimension $3(2k+1)$.
- The eigenspace corresponding to spectrum of $\nabla_u E_\mu(u)$ in first case of equation (11) is $\mathcal{V}_k \otimes \mathcal{W}_4$.

Computation of Bifurcation index $\omega_G(\mu_{km})$

Notice that

- $\omega_G(\mu_{km}) := \nabla_G\text{-deg}\,(\nabla_u^2 E_{\mu_-}(u), \mathcal{U}) - \nabla_G\text{-deg}\,(\nabla_u^2 E_{\mu_+}(u), \mathcal{U})$.
-

$$\begin{aligned}\omega_G(\mu_{km}) &= \prod_{\{(j,n)\in\mathbb{N}^2 : s_{jn} < s_{km}\}} \nabla_G\text{-deg}_{\mathcal{V}_j\otimes\mathcal{W}_4} - \prod_{\{(j,n)\in\mathbb{N}^2 : s_{jn} \le s_{km}\}} \nabla_G\text{-deg}_{\mathcal{V}_j\otimes\mathcal{W}_4}\\ &= \left(\prod_{\{(j,n)\in\mathbb{N}^2 : s_{jn} < s_{km}\}} \nabla_G\text{-deg}_{\mathcal{V}_j\otimes\mathcal{W}_4}\right)\left((G) - \prod_{\{(j,n)\in\mathbb{N}^2 : s_{jn} = s_{km}\}} \nabla_G\text{-deg}_{\mathcal{V}_j\otimes\mathcal{W}_4}\right)\\ &=: a * b. \qquad (17)\end{aligned}$$

* The basic degrees are invertible elements in $U(G)$.
* Let a, $b \in U(G)$ be such that a is an invertible element in $U(G)$ and $(H) \in \max(b)$. Then $\text{coeff}^H(a * b) \neq 0$.

Bifurcation in General Case

For certain S_{km}, to compute $b := \left((G) - \prod_{\{(j,n)\in\mathbb{N}^2 : s_{jn}=s_{km}\}} \nabla_G\text{-deg}_{\mathcal{V}_j\otimes\mathcal{W}_4} \right)$, we consider two cases:

(i) there is only one $n_0 \in \mathbb{N}^+$ such that $(0, n_0) \in \left\{(j, n) \in \mathbb{N}^2 : s_{jn} = s_{km}\right\}$

(ii) there is none.

For case (i),

$$\nabla_G\text{-deg}_{\mathcal{W}_4} = (G) - (O(3) \times D_2) - 2(O(3) \times D_3)) + 3(O(3) \times D_1) - (O(3)), \tag{18}$$

Therefore, we have

$$\begin{aligned} b :&= \nabla_G\text{-deg}_{\mathcal{W}_4} * \left(\nabla_G\text{-deg}_{\mathcal{W}_4} - \prod_{\{(j,n)\in\mathbb{N}^2 : s_{jn}=s_{km},\ j\neq 0\}} \nabla_G\text{-deg}_{\mathcal{V}_j\otimes\mathcal{W}_4} \right) \\ &= \nabla_G\text{-deg}_{\mathcal{W}_4} * c, \end{aligned} \tag{19}$$

where

$$c = -(O(3) \times D_2) - 2(O(3) \times D_3)) + 3(O(3) \times D_1) - (O(3)) + \sum_l \alpha_l (H_l),$$

and $\dim H_l = 0$ or 1.

Bifurcation in $\mathcal{V}_1 \otimes \mathcal{W}_4$

- Restriction of action of the group $S_4 < O(3)$ to the representation $\mathcal{V}_1$ is $\mathcal{W}_3$.
- Take the subgroup $G' := S_4^p \times S_4 \leq O(3) \times S_4, \quad S_4^p := S_4 \times \mathbb{Z}_2,$
- As a G'-representation, $\mathcal{V}_1 \otimes \mathcal{W}_4$ is equivalent to the irreducible $S_4^p \times S_4$-representation $\mathcal{W}_3^- \otimes \mathcal{W}_4$.

By using the GAP system one can compute the corresponding basic G'-equivariant degree

$$\begin{aligned}
\nabla_{G'}\text{-deg}_{\mathcal{W}_3^- \otimes \mathcal{W}_4} &= (G') - (D_4^{p\,\mathbb{Z}_2^-} \times_{D_4} D_4)_1 - (D_4^{p\,\mathbb{Z}_2^-} \times_{D_4} D_4)_2 - (D_4^{p\,D_4^z} \times^{D_2} D_4) \\
&- (D_4^{p\,D_4^z} \times^{D_1} D_2) - (D_3^{p\,D_3^z} \times^{D_1} D_2) - (D_3^{p\,D_3^z} \times^{D_2} D_4) \\
&- (D_2^{p\,D_2^d} \times^{D_2} D_4) - (D_2^{p\,D_2^d} \times^{D_1} D_2) - (D_4^z \times D_3) - (S_4^- \times_{S_4} S_4) \\
&- (D_2^d \times D_3) - (D_3^z \times D_3) - (D_3 \times_{D_3} D_3) + \alpha,
\end{aligned} \tag{20}$$

where α denotes the element in the Euler ring $U(G')$ with all the coefficients corresponding to sub-maximal orbit types.

Bifurcation in $\mathcal{V}_1 \otimes \mathcal{W}_4$

Notice that for a subgroup G' of G, i.e. $\psi : G_1 \to G$ is the natural embedding, induces the ring homomorphism $\Psi : U(G) \to U(G_1)$ called the *Euler ring homomorphism* and one has

$$\Psi\left(\nabla_G\text{-deg}_{\mathcal{V}_1 \otimes \mathcal{W}_4}\right) = \left(\nabla_{G'}\text{-deg}_{\mathcal{W}_3^- \otimes \mathcal{W}_4}\right). \tag{21}$$

$$\begin{aligned}\nabla_G\text{-deg}_{\mathcal{V}_1 \otimes \mathcal{W}_4} &= (G) - 2(D_4^{p^{\mathbb{Z}_2^-}} \times_{D_4} D_4) - (D_4^{p^{D_4^z}} \times^{D_2} D_4) \\ &- (D_4^{p^{D_4^z}} \times^{D_1} D_2) - (D_3^{p^{D_3^z}} \times^{D_1} D_2) - (D_3^{p^{D_3^z}} \times^{D_2} D_4) \\ &- (D_2^{p^{D_2^d}} \times^{D_2} D_4) - (D_2^{pD_2^d} \times^{D_1} D_2) - (D_4^z \times D_3) - (S_4^- \times_{S_4} S_4) \\ &- (D_2^d \times D_3) - (D_3^z \times D_3) - (D_3 \times_{D_3} D_3) + \beta,\end{aligned} \tag{22}$$

where β denotes the element in the Euler ring $U(G)$ with all the coefficients corresponding to sub-maximal orbit types.

Main Theorem

- Suppose that BECs is *immiscible* , i.e. $0 < g < \tilde{g}$.
- Denote by $\sigma(-\triangle) = \{s_{km}^2 : k, m \in \mathbb{N}\}$ the spectrum of the Laplace operator $-\triangle$ with Neumann boundary conditions in Ω
- Put critical value

$$\mu_{km} := \frac{g + 3\tilde{g}}{2\,(\tilde{g} - g)} s_{km}^2.$$

Theorem

- *The equation $\nabla_u E(u) = 0$ undergoes a global bifurcation from the trivial solution u_μ at any critical value μ_{km} with $(k, m) \neq (0, 0)$.*
- *In the case $k = 1$, there exists at least 13 G-orbits of global branches bifurcating from u_μ at the isotypic simple critical value μ_{1m} for $m \in \mathbb{N}^+$.*