Existence of Non-Radial Solutions to Laplace Systems on a Unit Ball in R^3

Jingzhou Liu

University of Texas at Dallas

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Problem Set Up

In this work, we study the existence of non-radial solutions to the following non-variational Laplace system on the unit ball $\Omega := \{x \in \mathbb{R}^3 : |x| < 1\}$:

$$\boxed{\begin{cases} -\triangle u = f(x, u), \quad u(x) \in \mathbb{R}^s, \\ u|_{\partial\Omega} = 0, \end{cases}} \tag{1}$$

where $f : \overline{\Omega} \times \mathbb{R}^s \to \mathbb{R}^s$ is a continuous odd radially symmetric function of sublinear growth, which is differentiable at zero.

We have the following assumptions:

(A_1) $f(gx, u) = f(x, u)$ for all $x \in \Omega$, $u \in \mathbb{R}^s$ and $g \in O(3)$;

(A_2) $f(x, -u) = -f(x, u)$ for all $x \in \Omega$, $u \in \mathbb{R}^s$;

(A_3) there exists a $s \times s$-matrix A, $c > 0$ and $\beta > 1$ such that $|f(x, u) - Au| \leq c|u|^\beta$ for all $x \in \overline{\Omega}, u \in \mathbb{R}^s$;

Problem Set Up

(A_4) there exist $a, b > 0$ and $\alpha \in (0,1)$ such that $|f(x,u)| < a|u|^\alpha + b \quad$ for all $\quad x \in \overline{\Omega}, u \in \mathbb{R}^s$.

(A_5) Matrix A is diagonalizable with eigenvalues $\mu_1 < \mu_2 < ... < \mu_s$

(A_6) for all $l = 1, 2, ..., s$, $m \in N$, $k = 0, 1, 2...$, one has $s_{km} \neq \mu_l$, where s_{km} denotes the m-th positive zero of the Bessel function $J_{k+1/2}$.

Notice that

* A_1, A_2 shows the system is $G := O(3) \times \mathbb{Z}_2$ symmetric.
* A_3 provides linearization of (1) at zero.
* A_4 is for reformulation of (1) in suitable functional space and obtaining *a priori* bounds.
* A_5, A_6 determines properties of the spectrum of linearization at zero.

Notations and Main Result

We apply the Brouwer G-equivariant degree theory to obtain the system (1) and the existence of various types of non-radial solutions. For this purpose, the following introduces essential notations.

- Denote by $\mathcal{V}_k$, $k = 0, 1, 2, \dots$ the complete list of the natural irreducible $O(3)$-representations (i.e. $O(3)$ acts on spherical harmonics of order k)
- Denote by $\mathcal{V}_k^- := \mathcal{V}_k$, $k = 0, 1, 2, \dots,$ the irreducible $O(3) \times \mathbb{Z}_2$-representations with the antipodal $\mathbb{Z}_2$-action.
- For a G-space X, $x \in X$, put $G_x := \{g \in G : gx = x\}$ and call it the *isotropy group* of x. Then the conjugacy class (G_x) is called the orbit type of x.
- We put $\Phi(G; X) := \{(G_x) : x \in X\}$ the set of all orbit types in X. Then for two conjugacy classes (H) and (K) of subgroups in G we write $(H) \leq (K)$ if there exists $g \in G$ such that $gHg^{-1} \leq K$.
- A conjugacy class (H) of a subgroup $H \leq G$ is called *maximal non-radial* if for some $k \in 2\mathbb{N} - 1$, (H) is a maximal element in $\Phi(G; \mathcal{V}_k \setminus \{0\})$ with respect to the relation '$\leq$'.

Notations and Main Result

Consider a non-radial maximal orbit type (H). Put

$$\mathscr{Z} := \{(k,m,l) : s_{km} < \mu_l,\ k \in 2\mathbb{N}-1,\ m \in \mathbb{N},\ l = 1,2,\dots,s\}.$$

Define the number $\mathfrak{m}_H$ to be the cardinality (which in our case is finite) of the set

$$\mathscr{Z}^H := \{(k,m,l) \in \mathscr{Z} : (H) \in \Phi(G;\mathcal{V}_k^-),\ \dim(\mathcal{V}_k^-)^H \text{ is odd}\}. \tag{2}$$

Then we have the following main result:

Under assumptions A1–A6, if (H) is non-radial maximal such that $\mathfrak{m}_H$ is odd, then system (1) admits a non-radial solution $u \in H_o^1(\Omega,\mathbb{R}^s) \cap H^2(\Omega,\mathbb{R}^s)$ such that $G_u \geq H$.

Functional Spaces Reformulation and *a priori* Bounds

- Consider Sobolev space $\mathscr{H} := H^1_o(\Omega, \mathbb{R}^s) \cap H^2(\Omega, \mathbb{R}^s)$ equipped with the Sobolev norm

$$\|u\|_{\mathscr{H}} := \max\{\|D^l u\|_{L^2} : |l| \le 2\}, \quad l = (l_1, l_2, l_3), \ |l| = l_1 + l_2 + l_3,$$

  where $D^l u = \frac{\partial^{|l|} u}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}}$.
- The linear operator $\mathscr{L} : \mathscr{H} \to L^2(\Omega; \mathbb{R}^s)$ defined by $\mathscr{L} u := -\triangle u, \quad u \in \mathscr{H}$ is an isomorphism.
- One can consider $\mathscr{L}$ as an unbounded operator

$$\mathscr{L} : D(\mathscr{L}) \subset L^2(\Omega; \mathbb{R}^s) \to L^2(\Omega; \mathbb{R}^s),$$

  which is clearly closed and self-adjoint unbounded Fredholm operator of index zero.
- By Poincaré inequality, the graph norm on $D(\mathscr{L})$ is equivalent with the Sobolev norm on $\mathscr{H}$, so the inverse operator $\mathscr{L}^{-1}$ is well defined and bounded.
- Choose $q > \max\{1, 2\alpha\}$, and let $j : \mathscr{H} \to L^q(\Omega; \mathbb{R}^s)$ be the standard Sobolev embedding. Then under the assumption A4, the function

$$N(v)(x) := f(x, v(x)), \quad x \in \overline{\Omega} \quad \text{for any} \quad v \in L^q(\Omega; \mathbb{R}^s), \tag{3}$$

  belongs to $L^2(\Omega, \mathbb{R}^s)$.
- The system (1) is equivalent to the equation $\mathscr{L} u = N(ju), \quad u \in \mathscr{H}$, which can also be written as $\mathscr{F}(u) = 0$, with the nonlinear operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$, given by

$$\mathscr{F}(u) := u - \mathscr{L}^{-1} N(ju) = 0, \quad u \in \mathscr{H}. \tag{4}$$

The following result provides *a priori* bounds for the system (1):

> Let $f : \overline{\Omega} \times \mathbb{R}^s \to \mathbb{R}^s$ be a continuous function satisfying the assumption A4, then there exists a constant $R > 0$ such that $\|u\|_{\mathscr{H}} < R$ for any solution $u \in \mathscr{H}$ to system (1).

Define the linear operator $\mathscr{A} : \mathscr{H} \to \mathscr{H}$, by

$$\mathscr{A}(u)(x) := u - \mathscr{L}^{-1} Au(x), \quad u \in \mathscr{H},\ x \in \overline{\Omega}.$$

Then, under the assumptions A3 and A4, the nonlinear operator $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ given by (4) is a completely continuous field differentiable at $0 \in \mathscr{H}$ with $D\mathscr{F}(0) = \mathscr{A}$.

Given the spectrum of $\mathscr{L}$:

$$\sigma(\mathscr{L}) = \{s_{km} : s_{km} \text{ is the } m\text{-th positive zero of } J_{k+\frac{1}{2}},\ k = 0, 1, 2, \dots\},$$

and its corresponding eigenspace $\mathscr{E}(s_{km})$ given by

$$\mathscr{E}(s_{km}) = \text{span}\left\{r^{-\frac{1}{2}} J_{k+\frac{1}{2}}(s_{km} r) P_k^n(\cos\varphi)\Big(a\cos(n\theta) + b\sin(n\theta)\Big) \quad a,\ b \in \mathbb{R},\ 0 \le n \le k\right\},$$

where

$$P_k^n(s) = \frac{(1-s^2)^{\frac{n}{2}}}{2^k k!} \frac{d^{k+n}}{ds^{k+n}}(s^2-1)^k$$

is called *Legendre Function. Then, we have that the G-isotypic decomposition of* $\mathscr{H}$ *is*

$$\mathscr{H} = \overline{\bigoplus_{k=0}^{\infty} \mathscr{H}_k},$$

where

$$\mathscr{H}_k := \overline{\bigoplus_{m=1}^{\infty} \mathscr{E}(s_{km})}.$$

Consequently, one has the following spectrum of the operator $\mathscr{A}$

$$\sigma(\mathscr{A}) = \left\{ \xi_{kml} := 1 - \frac{\mu_l}{s_{km}} : l = 1, 2, \dots, s,\ m \in \mathbb{N},\ k = 0, 1, 2, \dots \right\}.$$

The assumptions A4 and A5 imply that $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ is an isomorphism and each of the subspaces $\mathscr{E}(\xi_{kml})$ with eigenvalue ξ_{kml} of $\mathscr{A}$, is equivalent to the irreducible G-representation $\mathcal{V}_k^-$, i.e. in particular, $\mathscr{E}(\xi_{kml}) \subset \mathscr{H}_k$. We denote by $\sigma_-(\mathscr{A})$ the negative spectrum of $\mathscr{A}$, i.e.

$$\sigma_-(\mathscr{A}) := \{\xi_{kml} : s_{km} < \mu_l\}.$$

Proofs of the main results

For a maximal non-radial orbit type (H) in $\mathscr{H}$, we defined the number $\mathfrak{m}_H$ as the cardinality of the set $\mathscr{L}^H$. One should point out that maximal orbit types (H) in $\mathcal{V}_k^-$ belong to $\Phi_0(G) := \{(K) : K \leq G,\ W(K) \text{ is finite}\}$ and we have that

$$\deg_{\mathcal{V}_k} = (G) + n_H(H) + a,$$

where $\operatorname{coeff}^H(a) = 0$ and

$$n_H := \frac{(-1)^{\dim(\mathcal{V}_k^-)^H} - 1}{|W(H)|},$$

which implies that $n_H \neq 0$ if and only if $\dim(\mathcal{V}_k^-)^H$ is odd. Consequently, we obtain that

$$\mathfrak{m}_H := \left|\{\xi_{kml} \in \sigma_-(\mathscr{A}) : \operatorname{coeff}^H(\deg_{\mathcal{V}_k^-}) \neq 0\}\right| \tag{5}$$

Proofs of the main results

- $G := O(3) \times \mathbb{Z}_2$ acts naturally on the space $\mathscr{H}$ and $\mathscr{F}$ is G-equivariant.
- By assumption A5 and A6, $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ is an isomorphism.
- There exists an $\varepsilon > 0$ s.t. $\mathscr{F}$ is $B_\varepsilon(0)$-admissibly G-equivariantly homotopic to $\mathscr{A}$.
- $\mathscr{F}$ is $B_R(0)$-admissibly G-equivariantly homotopic to the identity Id.
- Put $\Omega := B_R(0) \setminus \overline{B_\varepsilon(0)}$ and $\mathscr{A}_k := \mathscr{A}|_{\mathscr{H}_k}$.

Then, by the additivity and product properties of the Brouwer G-equivariant degree,

$$\begin{aligned} G\text{-deg}(\mathscr{F}, \Omega) &= G\text{-deg}(\mathscr{F}, B_R(0)) - G\text{-deg}(\mathscr{F}, B_\varepsilon(0)) \\ &= G\text{-deg}(\mathrm{Id}, B_R(0)) - G\text{-deg}(\mathscr{A}, B_\varepsilon(0)) = (G) - G\text{-deg}(\mathscr{A}, B_\varepsilon(0)) \\ &= (G) - G\text{-deg}(\mathscr{A}_0, B(\mathscr{H}_0)) \cdot \prod_{k=1}^{\infty} \prod_{\xi_{kml} \in \sigma_-(\mathscr{A})} (\deg_{\mathcal{V}_k^-}) \end{aligned}$$

$$
\begin{aligned}
G\text{-deg}(\mathscr{F},\Omega) &= (G) - \big((G)-(O(3))\big)^s \cdot \prod_{k=1}^{\infty} \prod_{\xi_{kml}\in\sigma_-(\mathscr{A})} (\deg_{\mathcal{V}_k^-}) \quad \text{(for } s=0 \text{ or } 1)\\
&= (G) - \prod_{k=1}^{\infty} \prod_{\xi_{kml}\in\sigma_-(\mathscr{A})} (\deg_{\mathcal{V}_k^-}) + \boldsymbol{a}\\
&= (G) - \prod_{\xi_{kml}\in\sigma_-(\mathscr{A})} (\deg_{\mathcal{V}_k^-}) + \boldsymbol{a}\\
&= \begin{cases} \boldsymbol{b} & \text{if } \mathfrak{m}_H \text{ is even,}\\ n_H(H)+\boldsymbol{b} & \text{if } \mathfrak{m}_H \text{ is odd,} \end{cases}
\end{aligned}
$$

where $\boldsymbol{a}.\ \boldsymbol{b} \in A(G)$ satisfy $\text{coeff}^H(\boldsymbol{a}) = \text{coeff}^H(\boldsymbol{a}) = 0$ and

$$
n_H := \begin{cases} 1 & \text{if } |W(H)| = 2,\\ 2 & \text{if } |W(H)| = 1. \end{cases}
$$

- It follows from $n_H \neq 0$ that if $\mathfrak{m}_H$ is odd, then $G\text{-deg}(\mathscr{F},\Omega) \neq 0$. i.e. $\text{coeff}^H\left(G\text{-deg}(\mathscr{F},\Omega)\right) = n_H \neq 0$.
- By existence property of the Brouwer equivariant degree, there exists a non-zero solution u to the equation $\mathscr{F}(u) = 0$ such that $G_u \geq H$. This solution is not radial.

Maximal orbit types in $O(3) \times \mathbb{Z}_2$-representations $\mathcal{V}_k^-$

The following lists Table of $O(3) \times \mathbb{Z}_2$ maximal orbit types (H) in $\mathcal{V}_k$ with odd $d_k(H) := \dim((\mathcal{V}_k^-)^H)$, $k = 60m + l$ and $\frac{k}{2} < n \le k$:

l	m even — maximal (H) with odd $d_k(H)$	l	m odd — maximal (H) with odd $d_k(H)$
1	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	1	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
3	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	3	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
5	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	5	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
7	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	7	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
9	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}Z_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	9	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
11	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	11	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
13	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	13	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
15	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	15	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$ $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
17	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	17	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
19	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	19	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
21	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	21	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
23	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	23	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
25	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	25	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
27	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	27	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
29	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	29	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
31	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	31	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
33	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	33	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
35	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	35	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
37	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	37	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
39	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times Z_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	39	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
41	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	41	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_6}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
43	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	43	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
45	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	45	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
47	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	47	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
49	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	49	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
51	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	51	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
53	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	53	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
55	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	55	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
57	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	57	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
59	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$	59	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(A_5\times\mathbb{Z}_2)^{A_5}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{2n}\times\mathbb{Z}_2)^{D_{2n}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4}\times_{\mathbb{Z}_2}\mathbb{Z}_2$

Example

Let 2×2 matrix $A = \begin{bmatrix} 16 & 0 \\ 0 & 12 \end{bmatrix}$
We have two eigenvalues $\mu_1 = 12 < \mu_2 = 16$ of A.
The negative spectrum $\sigma_-(\mathscr{A}) := \{\xi_{kml} := 1 - \frac{\mu_l}{s_{km}} : \xi_{kml} < 0\}$ can be easily identified:

s_{km} k \ m	1	2	3	4	5	6	7	8	9
1	4.493	7.725	10.904	14.066	17.221	20.371	23.519	26.666	29.812
2	5.763	9.095	12.323	15.515	18.689	21.854	25.013	28.168	31.32
3	6.988	10.417	13.698	16.924	20.122	23.304	26.477	29.643	32.804
4	8.183	11.705	15.04	18.301	21.525	24.728	27.916	31.094	34.265
5	9.356	12.967	16.355	19.653	22.905	26.128	29.333	32.525	35.708
6	10.513	14.207	17.648	20.983	24.263	27.508	30.73	33.937	37.132
7	11.657	15.431	18.923	22.295	25.603	28.87	32.111	35.333	38.541
8	12.791	16.641	20.182	23.591	26.927	30.217	33.477	36.715	39.936
9	13.916	17.839	21.428	24.873	28.237	31.55	34.829	38.082	41.318
10	15.033	19.026	22.663	26.143	29.535	32.871	36.168	39.438	42.688
11	16.145	20.204	23.887	27.401	30.821	34.179	37.496	40.783	44.046
12	17.25	21.374	25.101	28.65	32.097	35.478	38.814	42.117	45.395

$\mathcal{V}_1^-, \dots \mathcal{V}_{10}^-$ are contributing to the eigenspaces of $\mathscr{A}$ associated with the negative eigenvalues.

$\mathcal{V}_l^-$, when $l = 2, 4, 6, 8, 10$ are related to the radial solutions, we only focus on the representations $\mathcal{V}_1^-, \mathcal{V}_3^-, \dots, \mathcal{V}_9^-$.

Example

The corresponding maximal non-radial orbit types are listed in the table below.

k	$m=0$ / maximal (H) with odd $d_k(H)$
1	$((O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
3	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_6\times\mathbb{Z}_2)^{D_6^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
5	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_6\times\mathbb{Z}_2)^{D_6^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_8\times\mathbb{Z}_2)^{D_8^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
	$(D_{10}\times\mathbb{Z}_2)^{D_{10}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
7	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_8\times\mathbb{Z}_2)^{D_8^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
	$(D_{10}\times\mathbb{Z}_2)^{D_{10}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{12}\times\mathbb{Z}_2)^{D_{12}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{14}\times\mathbb{Z}_2)^{D_{14}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
9	$(O(2)\times\mathbb{Z}_2)^{O(2)^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(S_4\times\mathbb{Z}_2)^{S_4^-}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{10}\times\mathbb{Z}_2)^{D_{10}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
	$(D_{12}\times\mathbb{Z}_2)^{D_{12}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{14}\times\mathbb{Z}_2)^{D_{14}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$, $(D_{16}\times\mathbb{Z}_2)^{D_{16}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$
	$(D_{18}\times\mathbb{Z}_2)^{D_{18}^d}\times_{\mathbb{Z}_2}\mathbb{Z}_2$

Example

Then for each maximal orbit type (H_l) in these components $\mathcal{V}_k$, we determine the number $\mathfrak{m}_l$ and apply Theorem to determine the existence of non-radial solution. The following table summarizes these results:

Group	m_H
$H_1 = (O(2) \times \mathbb{Z}_2)^{O(2)^-} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_1 = 19$
$H_2 = (S_4 \times \mathbb{Z}_2)^{S_4^-} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_2 = 9$
$H_3 = (D_6 \times \mathbb{Z}_2)^{D_6^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_3 = 8$
$H_4 = (D_8 \times \mathbb{Z}_2)^{D_8^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_4 = 6$
$H_5 = (D_{10} \times \mathbb{Z}_2)^{D_{10}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_5 = 7$
$H_6 = (D_{12} \times \mathbb{Z}_2)^{D_{12}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_6 = 4$
$H_7 = (D_{14} \times \mathbb{Z}_2)^{D_{14}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_7 = 4$
$H_8 = (D_{16} \times \mathbb{Z}_2)^{D_{16}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_8 = 1$
$H_9 = (D_{18} \times \mathbb{Z}_2)^{D_{18}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2$	$\mathfrak{m}_9 = 1$

Example

Conclusion:

Let $f : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map satisfying assumptions A1–A4 with

$$A = \begin{bmatrix} 16 & 0 \\ 0 & 12 \end{bmatrix}.$$

Then, for the following non-radial maximal orbit types (H)

$$(O(2) \times \mathbb{Z}_2)^{O(2)^-} \times_{\mathbb{Z}_2} \mathbb{Z}_2, \quad (S_4 \times \mathbb{Z}_2)^{S_4^-} \times_{\mathbb{Z}_2} \mathbb{Z}_2,$$
$$(D_{10} \times \mathbb{Z}_2)^{D_{10}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2, \quad (D_{16} \times \mathbb{Z}_2)^{D_{16}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2,$$
$$(D_{18} \times \mathbb{Z}_2)^{D_{18}^d} \times_{\mathbb{Z}_2} \mathbb{Z}_2,$$

there exists a non-radial solution $u \in H_o^1(\Omega, \mathbb{R}^s) \cap H^2(\Omega, \mathbb{R}^s)$ such that $G_u \geq H$.

Appendix: Spectrum of $\mathscr{L}$

In order to describe spectrum of $\mathscr{L}$,we can consider a more general situation where $\Omega \subset \mathbb{R}^d$ is the open unit ball in $\mathbb{R}^d$ ($d \geq 3$)

$$\begin{cases} -\triangle u = \mathscr{L} u = \lambda u, \quad x \in \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{6}$$

And use the spherical coordinates $(r, \boldsymbol{\theta})$ in $\mathbb{R}^d$:

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{d-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\triangle_{S^{d-1}} u, \tag{7}$$

where $\triangle_{S^{d-1}}$ is spherical Laplacian.

Use separation of variables:

Let $u(r, \boldsymbol{\theta}) = R(r) \cdot T(\boldsymbol{\theta})$, which is substituted to the equation $-\triangle u = \lambda u$ leads to

$$\frac{r^2 R'' + (d-1) r R'}{R} + \lambda r^2 = -\frac{\triangle_{S^{d-1}}(T)}{T} = c \tag{8}$$

$$\begin{cases} r^2 R'' + (d-1) r R' + (\lambda r^2 - c) R = 0 \\ \triangle_{S^{d-1}}(T) = cT. \end{cases} \tag{9}$$

Appendix: Spectrum of $\mathscr{L}$

Let $c := k(k+d-2)$, $R(r) := r^{-\frac{d-2}{2}}\hat{R}(r)$, $\widetilde{R}(t) = R(\frac{r}{\sqrt{\lambda}})$

$$t^2\widetilde{R}''(t) + t\widetilde{R}'(t) + \left(t^2 - \left(k + \frac{d-2}{2}\right)^2\right)\widetilde{R}(t) = 0 \tag{10}$$

This is classical Bessel equation.

* The bounded at zero solution to (6) is: $\widetilde{R}(t) = J_{k+\frac{d-2}{2}}(t)$.
* Therefore, the solution to (5) is $r^{-\frac{d-2}{2}}J_{k+\frac{d-2}{2}}(\sqrt{\lambda}r)$.
* $u(r,\theta) = R(r)T(\boldsymbol{\theta})$ satisfies the Dirichlet condition if $R(1) = 0$, i.e. $J_{k+\frac{d-2}{2}}(\sqrt{\lambda}) = 0$.

Given $d = 3$, then the spectrum of $\mathscr{L}$ is given by

$$\boxed{\sigma(\mathscr{L}) = \{s_{km} : s_{km} \text{ is the } m\text{-th positive zero of } J_{k+\frac{1}{2}},\ k = 0, 1, 2, \dots\}.}$$

Appendix: Spectrum of $\mathscr{L}$

The m th positive zero of Bessel function of the first kind $J_{k+\frac{1}{2}}(s_{km}) = 0$**, where** $0 \leq k \leq 19, 1 \leq m \leq 20$

s_{km} (k \ m)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	3.142	6.283	9.425	12.566	15.708	18.850	21.991	25.133	28.274	31.416	34.558	37.699	40.841	43.982	47.124	50.266	53.407	56.549	59.690	62.832
1	4.493	7.725	10.904	14.066	17.221	20.371	23.519	26.666	29.812	32.956	36.101	39.244	42.388	45.531	48.674	51.817	54.96	58.102	61.245	64.387
2	5.763	9.095	12.323	15.515	18.689	21.854	25.013	28.168	31.32	34.47	37.619	40.767	43.914	47.06	50.206	53.351	56.496	59.64	62.784	65.928
3	6.988	10.417	13.698	16.924	20.122	23.304	26.477	29.643	32.804	35.961	39.116	42.27	45.421	48.571	51.72	54.869	58.016	61.163	64.309	67.455
4	8.183	11.705	15.04	18.301	21.525	24.728	27.916	31.094	34.265	37.432	40.594	43.754	46.911	50.066	53.219	56.371	59.522	62.672	65.821	68.97
5	9.356	12.967	16.355	19.653	22.905	26.128	29.333	32.525	35.708	38.884	42.054	45.221	48.384	51.545	54.703	57.86	61.015	64.169	67.321	70.473
6	10.513	14.207	17.648	20.983	24.263	27.508	30.73	33.937	37.132	40.319	43.499	46.673	49.844	53.011	56.174	59.336	62.496	65.653	68.81	71.965
7	11.657	15.431	18.923	22.295	25.603	28.87	32.111	35.333	38.541	41.739	44.929	48.112	51.289	54.463	57.633	60.8	63.964	67.127	70.287	73.446
8	12.791	16.641	20.182	23.591	26.927	30.217	33.477	36.715	39.936	43.145	46.345	49.537	52.723	55.904	59.08	62.253	65.422	68.59	71.754	74.917
9	13.916	17.839	21.428	24.873	28.237	31.55	34.829	38.082	41.318	44.539	47.749	50.951	54.145	57.333	60.516	63.695	66.87	70.042	73.212	76.379
10	15.033	19.026	22.663	26.143	29.535	32.871	36.168	39.438	42.688	45.921	49.142	52.353	55.556	58.752	61.942	65.127	68.308	71.486	74.66	77.832
11	16.145	20.204	23.887	27.401	30.821	34.179	37.496	40.783	44.046	47.292	50.525	53.745	56.957	60.161	63.358	66.55	69.737	72.921	76.1	79.277
12	17.25	21.374	25.101	28.65	32.097	35.478	38.814	42.117	45.395	48.654	51.897	55.128	58.348	61.561	64.765	67.964	71.158	74.347	77.532	80.713
13	18.351	22.537	26.307	29.889	33.363	36.767	40.122	43.442	46.734	50.006	53.26	56.501	59.731	62.952	66.164	69.37	72.57	75.765	78.955	82.142
14	19.448	23.693	27.506	31.121	34.621	38.047	41.421	44.757	48.064	51.349	54.615	57.866	61.106	64.335	67.555	70.768	73.974	77.175	80.372	83.563
15	20.54	24.844	28.697	32.344	35.871	39.319	42.712	46.065	49.386	52.684	55.961	59.223	62.472	65.71	68.938	72.158	75.371	78.579	81.781	84.978
16	21.629	25.989	29.883	33.561	37.114	40.584	43.996	47.365	50.701	54.011	57.3	60.573	63.831	67.077	70.314	73.541	76.761	79.975	83.183	86.386
17	22.715	27.129	31.062	34.772	38.35	41.842	45.272	48.657	52.008	55.331	58.632	61.915	65.183	68.438	71.682	74.918	78.145	81.365	84.578	87.787
18	23.798	28.265	32.237	35.976	39.579	43.093	46.542	49.943	53.308	56.644	59.957	63.251	66.528	69.792	73.045	76.287	79.521	82.748	85.968	89.182
19	24.878	29.397	33.406	37.175	40.803	44.337	47.805	51.223	54.602	57.951	61.275	64.58	67.867	71.14	74.401	77.651	80.892	84.125	87.351	90.571

Appendix: Eigenspace of $\mathscr{L}$

Given $d = 3$,

* using the spherical coordinates (r, θ, ϕ), $r \geq 0$, $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi] \Rightarrow$

$$\triangle_{S^2} = \frac{\partial^2}{\partial \varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2} \tag{11}$$

* Let $T(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$, use seperation of variable to the equation $-\triangle_{S^2} T = cT$,

$$\frac{\sin^2\varphi \Phi'' + \cos\varphi \sin\varphi \Phi'}{\Phi} + c\sin^2\varphi = -\frac{\Theta''}{\Theta} = n^2, \tag{12}$$

where $n = 0, 1, \ldots$.

$$\begin{cases} \Theta(\theta) = a\cos(n\theta) + b\sin(n\theta) \\ \sin^2\varphi \Phi'' + \cos\varphi\sin\varphi\Phi' + \Big(k(k+1)\sin^2\varphi - n^2\Big)\Phi = 0. \end{cases} \tag{13}$$

Appendix: Eigenspace of $\mathscr{L}$

Let $s = \cos\varphi$, $P(s) = P(\cos\varphi) = \Phi(\phi)$, (9) becomes the classical Legendre equation:

$$(1 - s^2)P'' - 2sP' + \left(k(k+1) - \frac{n^2}{1 - s^2}\right) P = 0, \tag{14}$$

which, for $k \in \mathbb{Z}$, admits a bounded solution P_k^n, the *Legendre function* on $[-1, 1]$,

$$P_k^n(s) = \frac{(1 - s^2)^{\frac{n}{2}}}{2^k k!} \frac{d^{k+n}}{ds^{k+n}}(s^2 - 1)^k,$$

Eigenfunctions of $-\triangle_{S^2}$, corresponding to the eigenvalue $c = k(k+1)$, $k \geq 0$ are:

$$T_k^n(\theta, \varphi) := P_k^n(\cos\varphi)\Big(a\cos(n\theta) + b\sin(n\theta)\Big) \quad a,\ b \in \mathbb{R},\ 0 \leq n \leq k.$$

* The eigenspace $\mathscr{E}(s_{km})$ of $\mathscr{L}$ corresponding to s_{km} is:

$$\boxed{\mathscr{E}(s_{km}) = \operatorname{span}\left\{r^{-\frac{1}{2}} J_{k+\frac{1}{2}}(s_{km}r) T_k^n(\theta, \varphi) : a,\ b \in \mathbb{R},\ 0 \leq n \leq k\right\}.}$$

Appendix: Irreducible $O(3) \times \mathbb{Z}_2$ Representation and Isotypic Decomposition

Facts:

- The $O(3)$ representations $\mathcal{V}_k, k = 0, 1, 2, \ldots,$ which are called natural irreducible representations, are absolutely irreducible.
- he $O(3) \times \mathbb{Z}_2$-representation $\mathcal{V}_k^- = \mathcal{V}_k$ (with antipodal $\mathbb{Z}_2$-action) is also absolutely irreducible.
- The eigenspace $\mathscr{E}(s_{km})$ is $O(3) \times \mathbb{Z}_2$-equivalent to $\mathcal{V}_k^-$ (for every $m \in \mathbb{N}$)

The twisted subgroups of $SO(3) \times \mathbb{Z}_2$, which are

$$H^{\varphi} := \{(g, z) \in SO(3) \times \mathbb{Z}_2 : \varphi(g) = z\},$$

where $H \leq SO(3)$ and $\varphi : H \to \mathbb{Z}_2$ is a homomorphism.

* The subgroups H of $SO(3)$: $SO(3)$, $O(2)$, $SO(2)$, D_n, $n \geq 2$, $\mathbb{Z}_n$, $n \geq 1$, and the exceptional groups A_4, S_4 and A_5 , can be identified with H^{φ} where $\varphi : H \to \mathbb{Z}_2$ is a trivial homomorphism.

* Denote by $z : D_n \to \mathbb{Z}_2$ the epimorphism satisfying $\mathrm{Ker}\,(z) = \mathbb{Z}_n$,

* Denote by $d : D_{2n} \to \mathbb{Z}_2$, the epimorphism with $\mathrm{Ker}\,(d) = D_n$;

* Denote by $-$ the epimorphism $\varphi : S_4 \to \mathbb{Z}_2$ with $\mathrm{Ker}\,(\varphi) = A_4$;

* Denote by $-$ the epimorphism $\varphi : O(2) \to \mathbb{Z}_2$ with $\mathrm{Ker}\,(\varphi) = SO(2)$;

* Denote by $-$ the epimorphism $\varphi : O(3) \to \mathbb{Z}_2$ with $\mathrm{Ker}\,(\varphi) = SO(3)$

Subgroups $\mathscr{H} \leq G_1 \times G_2$:

There exist subgroups $H \leq G_1$ and $K \leq G_2$, a group L, and two epimorphisms $\varphi : H \to L$ and $\psi : K \to L$

$$\mathscr{H} = \{(h, k) \in H \times K : \varphi(h) = \psi(k)\}.$$

The notation used to describe $\mathscr{H}$ is

$$\mathscr{H} := H\,{}^{\varphi}\!\times_L^{\psi} K, \tag{15}$$

In our case, one can identify φ and ψ by their kernels $H_o := \mathrm{Ker}\,(\varphi)$ and $K_o := \mathrm{Ker}\,(\psi)$, i.e. the group $\mathscr{H}$ is the amalgamated notation can be written as

$$\mathscr{H} = H^{H_o} \times_L^{K_o} K.$$